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$$a \int_{\frac{1}{3}a}^{\frac{1}{2}a} \int_{\frac{1}{2}(3x-a)}^{\frac{1}{2}a} dx dy = \frac{2}{3}ax^2 = u_{11} = \frac{1}{48}a^3.$$

As the whole number of different trials that can be made will be represented by $8a^3$, if P represent the required probability, because the stable positions above calculated are just half the whole number of stable positions, (x having been taken only from the center of (1) to one extremity,) we shall have

$$P = \frac{u_1 + u_2 + \dots + u_{11}}{4a^3} = \frac{209}{2304}.$$

SOLUTION OF TWO PROBLEMS IN SUMMATION OF SERIES.

BY PROF. D. TROWBRIDGE, WATERBURGH, N. Y.

1. REQUIRED the sum of the series

$$1 + \frac{1}{1+r} + \frac{1}{1+2r} + \frac{1}{1+3r} + \dots + \frac{1}{1+(n-1)r} = \Sigma_n^{(r)} \dots (1)$$

Solution.—First find the value of the definite integral

$$\begin{aligned} \frac{1}{r} \int_{-1}^0 y^p dy (1+y)_r^{\frac{1}{r}-1} &= \left[y^p (1+y)_r^{\frac{1}{r}} - \frac{py^{p-1}r(1+y)^{(1+\frac{1}{r})-1}}{1+r} \right. \\ &+ \left. \frac{p(p-1)y^{p-2}r^2(1+y)^{(1+\frac{1}{r})+2}}{(1+r)(1+2r)} + (-1)^p \cdot \frac{p(p-1) \dots 2.1.r^p(1+y)^{(1+\frac{1}{r})+p}}{(1+r)(1+2r) \dots (1+pr)} \right] \\ &= (-1)^p \cdot \frac{p(p-1) \dots 2.1.r^p}{(1+r)(1+2r) \dots (1+pr)} = V_p \dots (2) \end{aligned}$$

Make $p = 0, 1, 2, 3$, &c., successively, and we shall have

$$\begin{aligned} V_0 &= 1, \quad V_1 = -\frac{r}{1+r}, \quad V_2 = \frac{2r^2}{(1+r)(1+2r)}, \\ V_3 &= -\frac{3.2r^3}{(1+r)(1+2r)(1+3r)}, \quad \&c. \dots (3) \end{aligned}$$

Now take the series

$$1 + x^r + x^{2r} + \dots + x^{(n-1)r} = \frac{x^{nr} - 1}{x^r - 1}, \quad \text{then}$$

$$\Sigma_n^{(r)} = \int_0^1 dx (1 + x^r + x^{2r} + \dots) = \int_0^1 dx \cdot \frac{x^{nr} - 1}{x^r - 1}.$$

Put $x^r = 1 + y$, $x = \sqrt[r]{1+y}$, $dx = (1+\frac{1}{r}) \times^{-1} \sqrt[r]{1+y} dy$, and

$$\frac{x^{nr} - 1}{x^r - 1} = \frac{(1+y)^n - 1}{y} = n + \frac{n(n-1)y}{1 \cdot 2} + \frac{n(n-1)(n-2)y^2}{1 \cdot 2 \cdot 3} + \dots$$

$$\Sigma_n^{(r)} = \int_{-1}^0 \frac{n}{r} \left[(1+y)^{\frac{1}{r}-1} + \frac{(n-1)y}{1 \cdot 2} (1+y)^{\frac{1}{r}-1} + \&c. \right] dy. \quad \dots \text{by (2)\&(3)}$$

$$\Sigma_n^{(r)} = n - \frac{n(n-1)r}{1 \cdot 2(1+r)} + \frac{n(n-1)(n-2)r^2}{1 \cdot 2 \cdot 3 \cdot (1+r)(1+2r)} - \frac{n(n-1)(n-2)(n-3)r^3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot (1+r)(1+2r)(1+3r)} + \dots (4)$$

$$\Sigma_n^{(1)} = n - \frac{n(n-1)}{1 \cdot 2 \cdot 2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 2 \cdot 3} - \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 2 \cdot 3 \cdot 4} + \dots (5)$$

From (4) we easily find

$$\begin{aligned} \Sigma_{n+1}^{(r)} - \Sigma_n^{(r)} &= \frac{1}{1+nr} = 1 - \frac{nr}{1+r} + \frac{n(n-1)r^2}{1 \cdot 2(1+r)(1+2r)} \\ &\quad - \frac{n(n-1)(n-2)r^3}{1 \cdot 2 \cdot 3(1+r)(1+2r)(1+3r)} + \dots \\ \frac{1}{1-nr} &= 1 + \frac{nr}{1+r} + \frac{n(n-1)r^2}{1 \cdot 2 \cdot (1+r)(1+2r)} + \frac{n(n-1)(n-2)r^3}{1 \cdot 2 \cdot 3(1+r)(1+2r)(1+3r)} + \dots \\ \frac{1}{1-n^2r^2} &= 1 + \frac{n(n-1)r^2}{1 \cdot 2(1+r)(1+2r)} + \frac{n(n-1)(n-2)(n-3)r^4}{1 \cdot 2 \cdot 3 \cdot 4(1+r)(1+2r)(1+3r)(1+4r)} + \dots \end{aligned}$$

2. Find the sum of the series

$$S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{n^2}. \quad \dots \dots \dots (6)$$

Since $1 + x + x^2 + x^3 + \dots + x^{n-1} = x^{n-1} \div (x-1)$ we evidently have

$$S_n = \int_0^1 \frac{dx}{x} \int_0^x \frac{dx}{x} \frac{x^n-1}{x-1}. \quad \dots \dots \dots (7)$$

Put $x-1=y$, $dx=dy$, then

$$\begin{aligned} \frac{x^n-1}{x-1} &= \frac{(1+y)^n-1}{y} = n + \frac{n(n-1)y}{1 \cdot 2} + \frac{n(n-1)(n-2)y^2}{1 \cdot 2 \cdot 3} + \dots \\ \int_0^x \frac{dx}{x} \frac{x^n-1}{x-1} &= \int_{-1}^y dy \left[n + \frac{n(n-1)y}{1 \cdot 2} + \dots \right] = ny + \frac{n(n-1)}{1 \cdot 2 \cdot 2} y^2 \\ &+ \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 3} y^3 + \dots + n - \frac{n(n-1)}{1 \cdot 2 \cdot 2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 3} + \dots \\ &= n(y+1) + \frac{n(n-1)}{1 \cdot 2 \cdot 2} (y^2-1) + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 3} (y^3+1) + \dots \end{aligned}$$

Therefore, since $dx \div x = dy \div (1+y)$

$$\begin{aligned} S_n &= \int_{-1}^0 dy \left[n + \frac{n(n-1)}{1 \cdot 2 \cdot 2} (y-1) + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 3} (y^2-y+1) + \dots \right] = n \\ &- \frac{n(n-1)}{1 \cdot 2 \cdot 2} (1+\frac{1}{2}) + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 3} (1+\frac{1}{2}+\frac{1}{3}) - \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 4} (1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}) + \dots \\ \therefore S_n &= n - \frac{n(n-1)}{1 \cdot 2 \cdot 2} \Sigma_2^{(1)} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 3} \Sigma_3^{(1)} - \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 4} \Sigma_4^{(1)} + \dots (8) \end{aligned}$$